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DECOMPOSITIONS AND THE RELATIVE TUBULAR NEIGHBOURHOOD CONJECTURE

C. P. ROURKE and B. J. SANDERSON

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THIS is a sequel to "Block bundles: II, transversality" [8]. We expand some of the statements made in §6 of [8] and prove, as announced there, that $B\widetilde{PL}_{n+q,n,q} \simeq G/PL$ for $n, q \geq 3$. This makes it easy to construct decompositions which are not block decompositions and hence counterexamples to the various conjectures we listed (see also Hudson [4] and Lickorish–Rourke [5]). As an interesting corollary of the existence of such examples we prove that there is no strong relative tubular neighbourhood theorem in the PL case; precisely, we give a counterexample (for all types of bundle) to the following:

Relative tubular neighbourhood conjecture:

Suppose $N \subset M \subset Q$ are proper submanifolds and ξ, η are normal (block, disc, or micro-) bundles on M in Q, which agree when restricted to N. Then there is an isotopy of $E(\xi)$ in Q mod $M \cup E(\xi \mid N)$ realising an isomorphism $\xi \cong \eta$.

Our construction works even in codimensions where Haefliger and Wall's results [3] hold and normal disc and microbundles exist uniquely!

We also have some information on $B\widetilde{PL}_{n+q,n,q}$ when n and q are not both ≥ 3.

Notation and definitions are as in [8].

§1. DECOMPOSITIONS AND BLOCK DECOMPOSITIONS

Recall (p. 276 of [8]) that $\widetilde{PL}^{n,q}_{n+q}$ is the Δ-group with typical k-simplex an isomorphism of the trivial decomposition $\varepsilon^{n+q}_{n,q}/\Delta^k$ with itself. It is the structural group for a decomposition of an $n+q$-block bundle into an n- and a q-dimensional factor. Restriction to the factors determines a homomorphism.

$$\varphi : \widetilde{PL}^{n,q}_{n+q} \to \widetilde{PL}_n \times \widetilde{PL}_q$$

and the kernel is denoted $\widetilde{PL}_{n+q,n,q}$. It is easy to see that φ is the projection of a principal bundle in the sense of [11; §1] with fibre $\widetilde{PL}_{n+q,n,q}$. A Δ-map $\alpha : \widetilde{PL}_n \times \widetilde{PL}_q \to \widetilde{PL}^{n,q}_{n+q}$ is defined by $\alpha(\sigma, \tau) = (\sigma \times 1) \circ (1 \times \tau)$ and $\varphi\alpha = 1$; thus α is a section for φ and hence the bundle is trivial and the Δ-map $\psi : \widetilde{PL}_{n+q,n,q} \times \widetilde{PL}_n \times \widetilde{PL}_q \to \widetilde{PL}^{n,q}_{n+q}$ given by $\psi(p, \sigma, \tau) = p \circ \alpha(\sigma, \tau)$ is an isomorphism of Δ-sets. However ψ is not a homomorphism since α is not.

Now the fibration

$$\widetilde{PL}_{n+q,n,q} \subset \widetilde{PL}^{n,q}_{n+q} \underset{\varphi}{\overset{\alpha}{\rightleftarrows}} \widetilde{PL}_n \times \widetilde{PL}_q \tag{1}$$

induces a fibration (see [11])

$$\mathfrak{B}(\widetilde{PL}_{n+q,n,q}) \subset \mathfrak{B}(\widetilde{PL}^{n,q}_{n+q}) \underset{\mathfrak{B}(\varphi)}{\overset{\alpha'}{\rightleftarrows}} \mathfrak{B}(\widetilde{PL}_n) \times \mathfrak{B}(\widetilde{PL}_q) \cong \mathfrak{B}(\widetilde{PL}_n \times \widetilde{PL}_q) \tag{2}$$

where $\mathfrak{B}$ ( ) is the semisimplicial classifying functor of [11; §1].

We can replace $\mathfrak{B}(\widetilde{PL}_n)$ etc. by polyhedra (denoted $B\widetilde{PL}_n$ etc.) of the same homotopy type (see §2 of [7]) and hence assume that there are genuine classifying block bundles, decompositions etc. over $B\widetilde{PL}_n$, $B\widetilde{PL}^{n,q}_{n+q}$ etc. This enables us to define a homotopy section α' to (2): Let $\gamma^n/B\widetilde{PL}_n$, $\gamma^q/B\widetilde{PL}_q$ be the classifying bundles. Then $\gamma^n \times \gamma^q/B\widetilde{PL}_n \times B\widetilde{PL}_q$ determines a classifying map $\alpha' : B\widetilde{PL}_n \times B\widetilde{PL}_q \to B\widetilde{PL}^{n,q}_{n+q}$. It is clear that $B(\varphi) \circ \alpha' \simeq 1$. Further if we are given a polyhedron X and a pair of maps (ξ, η) to $B\widetilde{PL}_n$ and $B\widetilde{PL}_q$ then $\alpha' \circ (\xi \times \eta)$ determines the Whitney sum decomposition of $\xi \oplus \eta/X$.

It follows that the long exact homotopy sequence of (2) splits and hence that the obstructions which prevent a decomposition from being a block decomposition (which are obstructions to lifting a map over α') have for coefficients the groups $\pi_i(\widetilde{PL}_{n+q,n,q})$ $(\cong \pi_{i+1}(B\widetilde{PL}_{n+q,n,q}))$ as stated in [8; §6].

In fact, in nearly all cases we can do better than this, and the obstruction can be interpreted as a homotopy class of maps into $B\widetilde{PL}_{n+q,n,q}$—see §4.

§2. BLOCK HOMOTOPY EQUIVALENCES

We recall some results of [11; §3] (stated in 2.1 and 2.2 below), which we need here. See also Casson [1].

Suppose ξ^n, η^n/K are block bundles. A *block map* $f: \xi \to \eta$ is a map of associated sphere bundles $f: E(\dot{\xi}) \to E(\dot{\eta})$ such that $f(E(\dot{\xi} \mid \sigma)) \subset E(\dot{\eta} \mid \sigma)$ for each $\sigma \in K$. There is an obvious notion of homotopy of block maps. Denote by f_σ the restriction $f \mid E(\dot{\xi} \mid \sigma): E(\dot{\xi} \mid \sigma) \to E(\dot{\eta} \mid \sigma)$.

PROPOSITION 2.1. *A block map $f: \xi \to \eta$ is a block homotopy equivalence if and only if each f_σ is a homotopy equivalence.*

A block homotopy equivalence $t: \xi \to \varepsilon$ (ε denotes the trivial block bundle) is called a *block homotopy trivialisation*. Trivialisations $t_i: \xi_i \to \varepsilon$, $i = 0, 1$, are *equivalent* if there is an isomorphism $j: \xi_0 \to \xi_1$ so that

$$\begin{tikzcd}
E(\xi_0) \arrow[d, "j"'] \arrow[dr, "t_0"] & \\
E(\xi_1) \arrow[r, "t_1"'] & E(\varepsilon)
\end{tikzcd}$$

commutes up to block homotopy.

Let $\mathfrak{E}_q(K)$ denote the set of equivalence classes of block homotopy trivialisations of q-block bundles with base K.

Let $\tilde{G}_q$, $\widetilde{PL}_q(\Sigma)$ be the Δ-sets defined in [9, p. 435] and $\tilde{G}_q/\widetilde{PL}_q(\Sigma)$ the set of right cosets (this is a Kan Δ-set by the usual argument).

PROPOSITION 2.2. *There is a bijection*

$$\mathfrak{E}_q(K) \to [K, \tilde{G}_q/\widetilde{PL}_q(\Sigma)].$$

§3. THE HOMOTOPY TYPE OF $\widetilde{BPL}_{n+q,n,q}$

LEMMA. *Suppose $\xi^n, \eta^q \subset \zeta^{n+q}/K$ is a decomposition and that ξ' is the complementary bundle to ξ in ζ (see* [8; 5.1]). *Then (up to block homotopy) there is determined a block homotopy equivalence $E(\dot{\xi}') \to E(\dot{\eta})$.*

Proof. There are charts $h_\sigma: \sigma \times I^n \times I^q \to E(\zeta)$ *for* ζ, which restrict to charts for ξ, η for each $\sigma \in K$. Also there is a strong deformation retraction

$$d_\sigma: \sigma \times X \to (\partial\sigma \times X) \cup \sigma \times \{0\} \times \partial I^q$$

where $X = \partial(I^n \times I^q) - \partial I^n \times \{0\}$.
Using the charts h_σ and the retractions d_σ we have a retraction $r: E(\dot{\zeta}) - E(\dot{\xi}) \to E(\dot{\eta})$.
The composition

$$E(\dot{\xi}') \subset E(\dot{\zeta}) - E(\dot{\xi}) \to E(\dot{\eta})$$

is the required block homotopy equivalence. That it is determined up to block homotopy is clear.

Now, as in §1, we replace $\tilde{G}_q/\widetilde{PL}_q(\Sigma)$ by a polyhedron denoted $(G/PL)_q$ over which we have a genuine classifying block bundle with block homotopy trivialisation.

We define a map

$$\theta: \widetilde{BPL}_{n+q,n,q} \to (G/PL)_q$$

as follows. Let $\varepsilon^n, \varepsilon^q \subset \gamma^{n+q}$ be the classifying bundle over $\widetilde{BPL}_{n+q,n,q}$. Let ξ^q be the complementary bundle to ε^n in γ^{n+q} then the lemma gives us a block homotopy trivialisation $\xi^q \to \varepsilon^q$. The classifying map for this is the required map θ.

THEOREM. *θ is a homotopy equivalence if $n \geq 3$.*

Proof. We define a map $\psi: (G/PL)_q \to \widetilde{BPL}_{n+q,n,q}$ so that θ, ψ are inverse homotopy equivalences.

Definition of ψ. Let $\eta^q \to \varepsilon^q$ be the classifying block bundle with block homotopy trivialisation over $(G/PL)_q$ and let $h': \varepsilon^q \to \eta^q$ be an inverse. We seek to embed ε^q in $\varepsilon^n \oplus \eta^q$ so that ε^n, $\varepsilon^q \subset \varepsilon^n \oplus \eta^q$ forms a decomposition. The embedding of $\dot{\varepsilon}^q$ in $(\varepsilon^n \oplus \eta^q) - \dot{\varepsilon}^n$ is found inductively by using the Casson–Sullivan embedding theorem (see [2, 6, 12]) and the block homotopy equivalence $h'|: \dot{\varepsilon}^q \to \dot{\eta}^q$ (this uses codim $= n \geq 3$). The embedding $\varepsilon^q \subset \varepsilon^n \oplus \eta^q$ then comes by inductively taking the cone on the embedding on the boundary. That ε^n, $\varepsilon^q \subset \varepsilon^n \oplus \eta^q$ is a decomposition follows from [8; Proof of 4.1] by induction.

The classifying map for this decomposition is the required map ψ.

Now it is clear from construction that $\theta\psi \simeq id$ and $\psi\theta \simeq id$ by the uniqueness (up to isotopy) of the embeddings given by the embedding theorem.

COROLLARY. *$\widetilde{BPL}_{n+q,n,q} \simeq G/PL$ if $n, q \geq 3$*
$\widetilde{BPL}_{n+q,n,q} \simeq *$ if either of n or $q \leq 2$ and n, q not both 2.

Proof. $(G/PL)_q \simeq G/PL$ if $q \geq 3$ and $\simeq *$ if $q \leq 2$ (see [9: 1.10]). If either of n or $q = 1$ then $\widetilde{BPL}_{n+q,n,q} \simeq *$ by an easy collaring argument (left to the reader). The only case left is $n = q = 2$ which we excluded.

PROBLEM. *Determine the homotopy type of $\widetilde{BPL}_{4,2,2}$.*

§4. THE MAP Φ: $\widetilde{BPL}^{n,q}_{n+q} \to G/PL$

Let $\gamma^n, \gamma^q \subset \zeta^{n+q}$ be the classifying bundle over $\widetilde{BPL}^{n,q}_{n+q}$ and let ξ^q be the complementary bundle to γ^n in ζ^{n+q}. Then the lemma provides a block homotopy equivalence $q: \xi_\sigma \to \gamma^q$. We wish to replace q by a block homotopy trivialisation of a stable bundle by adding to both sides a stable inverse to γ^q. However, since the base is an infinite dimensional complex we have to proceed by induction over skeleta:

Let $q_i: \xi_i^q \to \gamma_i^q$ denote the restriction to the i-skeleton of $\widetilde{BPL}^{n,q}_{n+q}$ and let α^{N_i} be a (stable) inverse to γ_i^q. Then we may assume that α^{N_i+1} extends $\alpha^{N_i} \oplus$ trivial bundle.

We obtain, for each i, stably compatible block homotopy trivialisations

$$q_i \oplus id: \xi_i^q \oplus \alpha^{N_i} \to \varepsilon^{N_i+q}$$

and hence maps $\Phi_i: (\widetilde{BPL}^{n,q}_{n+q})_i \to G/PL$ such that Φ_{i+1} extends Φ_i up to homotopy. This defines a limit map Φ, as required.

Now since α^{N_i} is stably trivial over a subcomplex on which γ_i^n is trivial, the following diagram commutes up to homotopy

$$\begin{array}{ccc} \widetilde{BPL}_{n+q,n,q} & \xrightarrow{i} & \widetilde{BPL}^{n,q}_{n+q} \\ \downarrow \theta & & \downarrow \Phi \\ (G/PL)_q & \xrightarrow{\Sigma} & G/PL \end{array} \tag{3}$$

(Σ is stable suspension)

COROLLARY. *Except possibly in the case $n = q = 2$, a decomposition $\xi, \eta \subset \zeta/X$ determines a map $\kappa: X \to \widetilde{BPL}_{n+q,n,q}$ with the property that $\kappa \simeq *$ iff the decomposition is a block decomposition.*

Proof. For $n, q \geq 3$ this follows from diagram (3) and the fact that θ and Σ are both homotopy equivalences. In the other cases, the result says nothing.

§5. THE RELATIVE TUBULAR NEIGHBOURHOOD CONJECTURE

Let $I^n = [-1, +1]^n$, $S^{n-1} = \partial I^n$ and $D^n = [-2, +2]^n \subset R^n$. We give a counter-example for $S^1 \times \{0\} \subset S^1 \times D^3 \times \{0\} \subset S^1 \times D^6$. The reader can then construct many other such examples in any codimensions $q \geq 3$.

Let $\varepsilon_1^3, \varepsilon_2^3 \subset \zeta^6/S^2$ be the non-trivial decomposition determined by the generator of $\pi_2(\widetilde{BPL}_{6,3,3}) \cong \pi_2(G/PL) \simeq \mathbf{Z}_2$ (see [9]). Regard $S^1 \subset S^2$ as the equator and let $D^2_\pm$ be the resulting hemispheres of S^2. The restriction $\varepsilon_1, \varepsilon_2 \subset \zeta/S^1$ is trivial and so we can identify it with the triple $S^1 \times I^3 \times \{0\}, S^1 \times \{0\} \times I^3 \subset S^1 \times I^6$.

Now consider $\varepsilon_1{}^3, \varepsilon_2{}^3 \subset \zeta \mid D_-{}^2$. This is a trivial decomposition, hence a block decomposition, and thus by definition there is a block bundle $\xi_-/E(\varepsilon_1{}^3 \mid D_-{}^2)$ so that the blocks of $\varepsilon_2 \mid D_-{}^2$ and $\eta \mid D_-{}^2$ are unions of blocks of ξ_-; similarly find $\xi_+/E(\varepsilon_1 \mid D_+{}^2)$. We can assume that $\xi_\pm \mid S^1 \times I^3 \times \{0\}$ are defined over the same cell complex. Then extend these latter to normal bundles of $S^1 \times D^3 \times \{0\}$ in $S^1 \times D^6$ by [7; 4.3] and call then $\xi_\pm{}^1$. These are the required bundles.

Suppose $\xi_+{}^1$ is isotopic to $\xi_-{}^1$ mod $S^1 \times D^3 \times \{0\} \cup E(\varepsilon_2 \mid S^1)$ then this implies that their restriction to $S^1 \times I^3 \times \{0\}$ are isotopic fixing the same subsets. This implies that there is a bundle $\xi/E(\varepsilon_1)$ so that the blocks of $E(\varepsilon_2)$ are unions of blocks of ξ (glue ξ_+ to ξ_- by the finishing homeomorphism of the isotopy) and then the triple $E(\varepsilon_1), E(\varepsilon_2) \subset E(\xi)$ has the same germ (near S^2) as $\varepsilon_1, \varepsilon_2 \subset \zeta$. This implies that $\varepsilon_1, \varepsilon_2 \subset \zeta$ is a block decomposition by the uniqueness part of [8; 6.1], which is restated and proved at the end of this section. This is a contradiction.

Finally note that since all the bundles involved were trivial over $S^1 \times D^3 \times \{0\}$ we could have taken them to be disc or micro bundles if we wanted. Also note that the proof shows that $\xi_+{}^1, \xi_-{}^1$ are not even concordant keeping $\varepsilon_2 \mid S^1$ fixed. Also, by doubling, we could have assumed that M and Q were closed manifolds.

PROPOSITION. *Suppose $M_1, M_2 \subset Q$ are transverse proper sub manifolds meeting in N. Suppose $|K| = N$ is a cell decomposition of N. Then there are normal block bundles $\xi_1, \xi_2, \xi/K$ on N in M_1, M_2, Q so that $\xi_1, \xi_2 \subset \zeta$ is a decomposition and the isomorphism class of this decomposition is unique.*

Proof. Existence for $K =$ dual cell decomposition of some triangulation is proved exactly as in [7; 4.3]. That the resulting bundles form a decomposition follows from transversality. For general K we now use the results of [8; §4] on amalgamating and subdividing decompositions. Uniqueness for $K =$ handle decomposition again follows the proof of [7; 4.4]. However the key isotopy [7, p. 18] is now constructed in three parts—first move the blocks of ξ_1 then of ξ_2 and finally of ζ using the fact that the latter is a relative regular neighbourhood of the former two. Uniqueness for general K now follows by an analogue of [7; 4.1].

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Mathematics Institute
University of Warwick